

Homework Problems
to go with
Fundamentals of
Massive MIMO

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About this Document

This document contains homework problems to go with the textbook *Fundamentals of Massive MIMO* by T. L. Marzetta, E. G. Larsson, Hong Yang and Hien Q. Ngo (Cambridge University Press, 2016). A suggested syllabus for a graduate-level course on Massive MIMO is also provided.

This is a living document, and new editions may be released in the future. The latest version may be obtained from the following web address:

www.cambridge.org/Marzetta

The estimated difficulty level of each problem is marked with the following symbols: ①: easy, ②: normal, and ③: hard. Problems marked with © require numerical work on a computer. Literature references in this document refer to the bibliography in the book.

A detailed solution manual is available for instructors adopting the book. To obtain a copy of this solution manual, write to erik.g.larsson@liu.se.

Suggested Outline of a Course on Massive MIMO Fundamentals

The following suggested syllabus is inspired by a course given at Linköping University in 2016:

Week 1: Overview, myths and realities.

Reading: Chapter 1, and the papers [35,45]

Homework: 1.1, 1.2, 1.3, 1.4, 1.5

Week 2: Mathematical background – Gaussian variables and Wishart distributions.

Reading: Appendices A–B

Homework: 9.1, 9.2, 9.3, 9.4, 9.5, 9.7, 10.1, (9.6ab), (10.2), (10.3)

Week 3: Scalar, Point-to-Point MIMO, and Multiuser MIMO channels.

Reading: Sections 2.2, C.2.1, C.2.2, C.2.4, C.3, C.4. Additional reference material on Point-to-Point MIMO, and to some extent Multiuser MIMO, can be found in [29].

Homework: 11.1, 11.2, 11.3, 11.5, 11.6, 11.7, 11.8, 11.9, 11.16, 11.18, (11.17)

Week 4: Techniques for capacity bounding. Channel coherence.

Reading: Chapter 2 and Appendix C

Homework: 11.4, 11.10, 11.11, 11.12, 11.13, 11.14, 2.1, 2.2, 2.3, 2.4, (2.7), (2.5), (11.15)

Weeks 5–6: Analysis of single-cell Massive MIMO systems.

Reading: Chapter 3 and Appendix D

Homework: 3.1, 3.2, 3.3, 3.4, 3.6, 3.7, 3.10, 3.13, (3.11), (3.12), (2.6)

Week 7: Analysis of multi-cell Massive MIMO systems.

Reading: Chapter 4

Homework: 4.1, 4.2, (4.3), (4.4), (4.5)

Week 8: Power control.

Reading: Chapter 5

Homework: 5.1, 5.2, 5.3, 5.6, (5.4)

Week 9: Case studies: rural and mobile access.

Reading: Chapter 6

Homework: 6.1, 6.2, 6.3, 6.4, 6.5, (6.6)

Week 10: Propagation

Reading: Chapter 7 and the papers [42, 46, 47]

Homework: 7.2, 7.3, 7.4, 7.5, 7.6, 7.7, 7.8, 7.9, (7.13), (7.14), (14)

Additional lectures that treat advanced topics may be included: MMSE processing and correlated fading (based on, e.g., [30, 65] and Y. Liu, T. Wong, and W. Hager, “Training signal design for estimation of correlated MIMO channels with colored interference,” *IEEE Trans. Signal Process.*, vol. 55, no. 4, pp. 1486–1497, 2007) and hardware impairment effects (based on, e.g., [116, 118, 123]).

Problems to Chapter 1**1.1 Sylvester's identity.**^①

Prove Sylvester's determinant identity.

1.2 Capacity of Point-to-Point MIMO.^{②③}

Reproduce Figure 1.2, and include a curve for the case that \mathbf{G} is known at the base station.

1.3 TDD versus FDD.^①

Give equations corresponding to (1.5) and (1.6) for the interference limited case, and interpret these equations.

1.4 Constrained capacity.^②

Give a first-order estimate of how much the vertical line in Figure 1.6 would shift to the right if the modulation were constrained to QPSK.

Hint: Consider the AWGN channel with the input alphabet constrained to QPSK, in the relevant spectral efficiency regime.

1.5 Spatial division multiple access.^①

Read [1, 12–19] (though no need to study in depth). Discuss how the ideas in these references relate to Multiuser MIMO, or Massive MIMO (if they do). In what ways does Massive MIMO differ from traditional spatial division multiple access concepts?

Problems to Chapter 2

2.1 Channel coherence at 60 GHz.^①

Recompute the values in Table 2.1 for a carrier frequency of 60 GHz. Discuss the result.

2.2 Noise with nonzero mean.^①

Consider the scalar point-to-point channel, $y = \sqrt{\rho}x + w$, with a power constraint $\mathbb{E}\{|x|^2\} \leq 1$ and where $w \sim \text{CN}(1/\sqrt{2}, 1/2)$ is noise that is independent of x . Determine the capacity of this channel. More generally, discuss the effects of a nonzero noise mean on capacity.

2.3 Capacity with two superimposed symbols.^②

Consider a channel that takes (x_1, x_2) as input and yields y as output,

$$y = a_1x_1 + a_2x_2 + w, \quad (\text{H-1})$$

where a_1 and a_2 are channel gains, and $w \sim \text{CN}(0, 1)$ is noise. The symbols x_1 and x_2 have unit power and are independent of a_1 , a_2 , and w . Neither the transmitter, nor the receiver, knows a_1 or a_2 .

The capacity is

$$C = \max_{p_{x_1, x_2(\cdot, \cdot)}} I\{y; x_1, x_2\}. \quad (\text{H-2})$$

Show that

$$C \geq \log_2 \left(1 + \frac{|\mathbb{E}\{a_1\}|^2 + |\mathbb{E}\{a_2\}|^2}{\text{Var}\{a_1\} + \text{Var}\{a_2\} + 1} \right) \quad (\text{H-3})$$

2.4 Joint transmission (“network MIMO”).^②

Consider a system where two base stations jointly serve one terminal on the downlink. Each base station has M antennas and the terminal has a single antenna. Let \mathbf{g}_1 and \mathbf{g}_2 be the $M \times 1$ channels vectors between the two base stations and the terminal. The elements of \mathbf{g}_1 and \mathbf{g}_2 are i.i.d. $\text{CN}(0, 1)$.

The base stations know \mathbf{g}_1 and \mathbf{g}_2 perfectly and apply maximum-ratio processing to obtain their respective transmit signals. Hence, the precoded signals transmitted by base stations 1 and 2 are given by

$$\mathbf{x}_1 = \sqrt{\frac{1}{M}} \mathbf{g}_1^* q_1, \quad \mathbb{E}\{|q_1|^2\} = 1, \quad (\text{H-4})$$

$$\mathbf{x}_2 = \sqrt{\frac{1}{M}} \mathbf{g}_2^* q_2, \quad \mathbb{E}\{|q_2|^2\} = 1. \quad (\text{H-5})$$

The terminal receives

$$y = \sqrt{\rho_{\text{dl}}}\mathbf{g}_1^T \mathbf{x}_1 + \sqrt{\rho_{\text{dl}}}\mathbf{g}_2^T \mathbf{x}_2 + w, \quad (\text{H-6})$$

where ρ_{dl} is the downlink SNR, and w is $\text{CN}(0, 1)$ noise. The objective for the terminal is to detect q_1 and q_2 from y . Nominally the terminal does not know \mathbf{g}_1 or \mathbf{g}_2 .

- (a) If $q_1 = q_2$, we have *coherent* joint transmission. Show that the capacity is lower-bounded by

$$C_{\text{coh}} \geq \log_2 \left(1 + \frac{4M\rho_{\text{dl}}}{2\rho_{\text{dl}} + 1} \right). \quad (\text{H-7})$$

- (b) If q_1 and q_2 are independent, we have *non-coherent* joint transmission. Using the results of Exercise 3, show that capacity is lower-bounded by

$$C_{\text{noncoh}} \geq \log_2 \left(1 + \frac{2M\rho_{\text{dl}}}{2\rho_{\text{dl}} + 1} \right). \quad (\text{H-8})$$

- (c) Comment, quantitatively, on the difference between (H-7) and (H-8).
 (d) In (a)–(b), suppose the terminal, by the aid of a genie, knows $\|\mathbf{g}_1\|$ and $\|\mathbf{g}_2\|$. Give lower bounds on capacity that apply in this situation, and compare to (H-7) respectively (H-8).

2.5 Channel hardening facilitates simple terminal signal processing.^②

Consider the downlink with a single terminal and maximum-ratio processing at the base station. The terminal receives

$$y = \sqrt{\rho}\mathbf{g}^T \frac{\mathbf{g}^*}{\sqrt{M}}q + w = cq + w, \quad (\text{H-9})$$

where ρ is the SNR, \mathbf{g} is the channel vector comprising M $\text{CN}(0, 1)$ elements (the path loss is unity here for simplicity),

$$c = \sqrt{\rho}\mathbf{g}^T \frac{\mathbf{g}^*}{\sqrt{M}} \quad (\text{H-10})$$

is the gain of the effective channel seen by the terminal, q is the transmitted unit-power symbol, and $w \sim \text{CN}(0, 1)$ is noise.

- (a) Derive the capacity, assuming that the terminal, by the aid of a genie, perfectly knows c .
 (b) Suppose that the terminal, in the spirit of “use and forget CSI”, processes the received signal y by dividing it by the effective channel gain c , resulting in $y' = y/c$. Explain why the channel between q and y' is a deterministic channel with additive noise, and derive a corresponding capacity bound.

- (c) Analytically and numerically compare the expressions in (a) and (b) for $M = 2$ and $M = 20$. Comment on the result.

2.6 Downlink pilots.^②

Consider the downlink with an M -antenna base station and K single-antenna terminals. The $M \times 1$ channel vector between the k th terminal and the base station is denoted by \mathbf{g}_k . Assume that the elements of \mathbf{g}_k are i.i.d. $\text{CN}(0, 1)$ and that, by the aid of a genie, the base station perfectly knows the channels $\{\mathbf{g}_k\}$. With maximum-ratio processing, the base station transmits,

$$\mathbf{x} = \sqrt{\frac{\rho}{MK}} \sum_{k=1}^K \mathbf{g}_k^* q_k, \quad (\text{H-11})$$

where ρ is the SNR, and q_k is the unit-power symbol intended for the k th terminal. The k th terminal receives

$$y_k = \mathbf{g}_k^T \mathbf{x} + w_k = c_{kk} q_k + \sum_{k' \neq k} c_{kk'} q_{k'} + w_k, \quad (\text{H-12})$$

where

$$c_{kk'} = \sqrt{\frac{\rho}{MK}} \mathbf{g}_k^T \mathbf{g}_{k'}^*. \quad (\text{H-13})$$

- (a) Derive a capacity lower bound for the case when the k th terminal does not know c_{kk} , but knows its statistics, i.e., $\mathbf{E}\{c_{kk}\}$.
- (b) Suppose that the k th terminal, by the aid of a genie, perfectly knows $\{c_{kk'}\}$. Derive a corresponding capacity lower bound.
- (c) Assume that the k th terminal estimates the effective channel gain c_{kk} from downlink pilots. Let

$$\mathbf{S} = \sqrt{\frac{\tau_{d,p}\rho}{MK}} \sum_{k=1}^K \mathbf{g}_k^* \phi_k \quad (\text{H-14})$$

be the pilot matrix sent from the base station, where ϕ_k is a $\tau_{d,p} \times 1$ vector satisfying $\|\phi_k\|^2 = 1$, and $\phi_k^H \phi_{k'} = 0$, for $k \neq k'$. Then the received pilot vector at the k th terminal is

$$\mathbf{y}_{p,k} = \mathbf{g}_k^T \mathbf{S} + \mathbf{w}_{p,k} = \sqrt{\tau_{d,p}} c_{kk} \phi_k + \sqrt{\tau_{d,p}} \sum_{k' \neq k} c_{kk'} \phi_{k'} + \mathbf{w}_{p,k}. \quad (\text{H-15})$$

- (i) Find the linear MMSE estimate, \hat{c}_{kk} , of c_{kk} from the received pilot vector (H-15).

- (ii) Suppose the k th terminal processes its received signal (H-12) by dividing it by the channel estimate \hat{c}_{kk} , resulting in a deterministic channel with additive uncorrelated noise,

$$\begin{aligned} y'_k &= \frac{y_k}{\hat{c}_{kk}} = \frac{c_{kk}}{\hat{c}_{kk}} q_k + \sum_{k' \neq k}^K \frac{c_{kk'}}{\hat{c}_{kk}} q_{k'} + \frac{w_k}{\hat{c}_{kk}} \\ &= \mathbb{E} \left\{ \frac{c_{kk}}{\hat{c}_{kk}} \right\} q_k + \left(\frac{c_{kk}}{\hat{c}_{kk}} - \mathbb{E} \left\{ \frac{c_{kk}}{\hat{c}_{kk}} \right\} \right) q_k + \sum_{k' \neq k}^K \frac{c_{kk'}}{\hat{c}_{kk}} q_{k'} + \frac{w_k}{\hat{c}_{kk}}. \end{aligned} \quad (\text{H-16})$$

Use the technique in Section 2.3.2 to derive a capacity lower bound.

- (iii) Numerically compare the bounds in (a), (b), and (c) taking into account the pilot overhead, for $K = 5$, $T = 200$ and $\tau_{d,p} = K$. Plot the results for $M = 5$ and $M = 50$.

2.7 Capacity bound with side information.^②

A terminal receives the following signal:

$$y_1 = a_1 x_1 + \sum_{k=2}^K a_k x_k + w, \quad (\text{H-17})$$

where x_1 is a desired symbol, x_2, \dots, x_K are interfering symbols, a_1, \dots, a_K , are channel gains, and $w \sim \text{CN}(0, 1)$ is noise. All symbols have zero mean and unit power. Also, all symbols are mutually independent, and independent of $\{a_k\}$ and w . Furthermore, a_1 is independent of a_2, \dots, a_K .

- Derive a capacity lower bound for the case when the receiver perfectly knows a_1 .
- Derive a capacity lower bound for the case when the receiver perfectly knows a_1, \dots, a_K .
- Compare the bounds in (a) and (b).

Problems to Chapter 3

3.1 Variance calculations.^①

Prove (3.38), step by step.

3.2 Pilot sequence choices.^①

What is the relative performance of the following two choices of length- K pilot sequences:

- (a) $[\sqrt{K}, 0, \dots, 0]; [0, \sqrt{K}, 0, \dots, 0]; \dots [0, \dots, 0, \sqrt{K}]$.
- (b) The rows of a $K \times K$ Hadamard matrix (each element is $+1$ or -1).
- (c) The rows of a $K \times K$ DFT matrix, whose (k, k') th element is equal to $e^{-j2\pi kk'/K}$.

3.3 Flawed capacity bounding argument.^①

Consider (3.32). Assess the following claim:

“The signal power is $A = \rho_{\text{ul}} \gamma_k \eta_k \mathbb{E} \{ \|z_k\|^4 \}$. The noise power is the expected value of the second term, squared, say B . So the capacity is bounded by $\log_2(1 + A/B)$.”

Explain why the argument in this claim is incorrect.

3.4 Asymptotic properties of zero-forcing and maximum-ratio processing.^②

Consider the single-cell Massive MIMO uplink with i.i.d. Rayleigh fading.

- (a) Show that as $M \rightarrow \infty$ for fixed K , the outputs of the maximum-ratio and zero-forcing processing decoders, $\mathbf{A}^H \mathbf{y}$, appropriately normalized, become equal.
- (b) Against the background of (a), consider the gap between the capacity bounds for maximum-ratio and zero-forcing processing for large M in Figure 3.5. Explain why there is a significant gap, and work out a first-order approximation of the gap size.

3.5 Alternative capacity bounds.^②

Derive the effective SINR in (D.12).

3.6 Capacity bounds for scalar channel.^①

Consider the case of $M = 1, K = 1$, discussed at the beginning of Section 3.4. Give an appropriate capacity bound for this case.

3.7 Alternative precoder normalization.^③

Consider the single-cell Massive MIMO downlink with maximum-ratio processing, perfect CSI at the base station, and equal power allocation to all terminals.

(a) Show that the following expression for the transmitted vector,

$$\mathbf{x} = \sqrt{\frac{\rho_{\text{dl}}}{MK}} \left(\frac{\mathbf{g}_1^*}{\sqrt{\beta_1}} q_1 + \cdots + \frac{\mathbf{g}_K^*}{\sqrt{\beta_K}} q_K \right), \quad \mathbb{E} \{|q_k|^2\} = 1, \quad (\text{H-18})$$

is equivalent to what is used in the book. Also show that an achievable rate of the k th terminal is

$$R_{\text{long-term}} = \log_2 \left(1 + \frac{M}{K} \frac{\rho_{\text{dl}} \beta_k}{\rho_{\text{dl}} \beta_k + 1} \right). \quad (\text{H-19})$$

(b) In contrast, suppose we apply a short-term power normalization:

$$\mathbf{x} = \sqrt{\frac{\rho_{\text{dl}}}{K}} \left(\frac{\mathbf{g}_1^*}{\|\mathbf{g}_1\|} q_1 + \cdots + \frac{\mathbf{g}_K^*}{\|\mathbf{g}_K\|} q_K \right), \quad \mathbb{E} \{|q_k|^2\} = 1. \quad (\text{H-20})$$

Show that an achievable rate for the k th terminal in this case is

$$R_{\text{short-term}} = \log_2 \left(1 + \frac{\frac{\rho_{\text{dl}}}{K} \beta_k \left(\frac{\Gamma(M+1/2)}{\Gamma(M)} \right)^2}{\frac{\rho_{\text{dl}}}{K} \beta_k \left(M - \left(\frac{\Gamma(M+1/2)}{\Gamma(M)} \right)^2 \right) + \rho_{\text{dl}} \frac{K-1}{K} \beta_k + 1} \right). \quad (\text{H-21})$$

(c) Plot $R_{\text{long-term}}$ and $R_{\text{short-term}}$ for $K = 10$, $\rho_{\text{dl}} = 0$ dB, $\beta_k = 1$ for all k and different M . Discuss the result.

(d) Show that, even though both rates grow without bound as $M \rightarrow \infty$,

$$\frac{R_{\text{short-term}}}{R_{\text{long-term}}} \rightarrow 1, \quad \text{as } M \rightarrow \infty. \quad (\text{H-22})$$

3.8 Alternative precoder normalization.^③

Repeat Problem 3.7, but for zero-forcing processing.

3.9 Non-orthogonal pilots.^③

Consider single-cell Massive MIMO. In contrast to the book, assume that the pilot sequences of the K terminals are arbitrary, i.e., not necessarily mutually orthogonal. Derive uplink and downlink achievable rates for the k th terminal, for maximum-ratio processing. Discuss the result.

3.10 Successive interference cancellation.^②

Consider the single-cell Massive MIMO uplink as in the book, but with perfect CSI at the base station. Suppose every terminal expends its maximum permitted power. The signal received at the base station is then,

$$\mathbf{y} = \sqrt{\rho_{\text{ul}}} \sum_{k=1}^K \mathbf{g}_k q_k + \mathbf{w}. \quad (\text{H-23})$$

Suppose, further, that the base station uses maximum-ratio processing followed by successive interference cancellation to detect the signals q_k , $k = 1, \dots, K$, as follows: The base station first uses maximum-ratio processing to detect q_1 . Then $\sqrt{\rho_{\text{ul}}}\mathbf{g}_1 q_1$ is subtracted from \mathbf{y} . The resulting signal is used to detect q_2 by maximum-ratio processing, and so forth until q_K is detected.

Show that a lower bound on the sum-capacity for this scheme is given by

$$\begin{aligned} R_{\text{sum}}^{\text{succ}} &= \log_2 \left(1 + \frac{M\rho_{\text{ul}}\beta_1}{1 + \rho_{\text{ul}} \sum_{k'=1}^K \beta_{k'}} \right) + \log_2 \left(1 + \frac{M\rho_{\text{ul}}\beta_2}{1 + \rho_{\text{ul}} \sum_{k'=2}^K \beta_{k'}} \right) + \dots \\ &+ \log_2 \left(1 + \frac{M\rho_{\text{ul}}\beta_K}{1 + \rho_{\text{ul}}\beta_K} \right). \end{aligned} \quad (\text{H-24})$$

Compare sum rate in (H-24) with that in Table 3.2.

3.11 Correlated Rayleigh fading.^③

Consider the single-cell Massive MIMO uplink with maximum-ratio processing and correlated Rayleigh fading: the channel vectors $\mathbf{g}_k \sim \text{CN}(0, \mathbf{C}_k)$, where \mathbf{C}_k is an $M \times M$ covariance matrix. For simplicity, disregard power control; take all power control coefficients equal.

(a) Write the MMSE channel estimate of \mathbf{g}_k as

$$\hat{\mathbf{g}}_k = \mathbf{g}_k + \tilde{\mathbf{g}}_k, \quad (\text{H-25})$$

where $\tilde{\mathbf{g}}_k$ is the estimation error, independent of $\hat{\mathbf{g}}_k$. Show that $\hat{\mathbf{g}}_k$ and $\tilde{\mathbf{g}}_k$ are complex Gaussian with zero mean and covariance matrices,

$$\mathbf{R}_k = \tau_p \rho_{\text{ul}} \mathbf{C}_k (\tau_p \rho_{\text{ul}} \mathbf{C}_k + \mathbf{I}_M)^{-1} \mathbf{C}_k, \quad (\text{H-26})$$

respectively,

$$\mathbf{R}_{e,k} = \mathbf{C}_k - \mathbf{R}_k. \quad (\text{H-27})$$

(b) Use “use and forget CSI” capacity-bounding technique to show that

$$C_k \geq \log_2 \left(1 + \frac{\rho_{\text{ul}} (\text{Tr} \{ \mathbf{R}_k \})^2}{\rho_{\text{ul}} \sum_{k'=1}^K \text{Tr} \{ \mathbf{R}_k \mathbf{C}_{k'} \} + \text{Tr} \{ \mathbf{R}_k \}} \right). \quad (\text{H-28})$$

(c) Consider the following specific model for the fading correlation:

$$\mathbf{C}_k = \begin{bmatrix} 1 & r_k^* & \dots & (r_k^*)^{M-1} \\ r_k & 1 & \dots & (r_k^*)^{M-2} \\ \vdots & \vdots & \ddots & \vdots \\ r_k^{M-1} & r_k^{M-2} & \dots & 1 \end{bmatrix}, \quad (\text{H-29})$$

where $|r_k| \in [0, 1]$ and $\arg(r_k)$ is arbitrary.

- (i) Show that \mathbf{C}_k is positive semi-definite.
 Hint: Construct a fictitious, wide-sense stationary time-series (discrete-time random process), $\{x_n\}$, with autocorrelation function $\{r_k^n\}$, and such that \mathbf{C}_k is the covariance matrix of the snapshot $[x_1, \dots, x_M]^T$.
- (ii) Plot the right-hand side of (H-28) for $k = 1$ as function of ρ_{ul} for $M = 100$, $\tau_p = K = 20$, with $|r_k| = 0, 0.5$, and 0.9 and:
 - *Case 1*: $\arg(r_1) = 0$, and $\arg(r_k) = \frac{\pi}{2} + \frac{\pi}{10}u$, for $k = 2, \dots, K$.
 - *Case 2*: $\arg(r_1) = 0$, $\arg(r_2) = 0.1$, and $\arg(r_k) = \frac{\pi}{2} + \frac{\pi}{10}u$, for $k = 3, \dots, K$.

3.12 Massive MIMO versus grid-of-beams beamforming.^③

Consider the downlink with a single terminal. The channel vector between the terminal and the base station, \mathbf{g} , has M i.i.d. $\text{CN}(0, 1)$ elements.

- (a) Suppose that the base station, on a per-coherence interval basis, selects the best antenna, with index

$$m^* = \arg \max_m \{|g_m|^2\}, \quad (\text{H-30})$$

and uses only this antenna for the transmission. The received SNR at the terminal is proportional to $\text{E} \{|g_{m^*}|^2\}$. Show that this average SNR grows as $\ln(M)$ when M increases.

- (b) Suppose that the base station beamforms along one of M predetermined, mutually orthogonal channel vectors. Show that the average received SNR scales as $\ln(M)$ also in this case.
- (c) Give an example of a set of M mutually orthogonal channel vectors, in a line-of-sight propagation environment with a $\lambda/2$ -spaced uniform linear array.

3.13 MMSE processing.^②

Consider the single-cell Massive MIMO uplink where the signal received at the base station is given by \mathbf{y} in (3.18). Suppose that the base station processes this received signal through multiplication by a $K \times M$ decoding matrix \mathbf{A}^{H} that is a function of the channel estimate $\hat{\mathbf{G}}$. The k th component of the processed signal is

$$[\mathbf{A}^{\text{H}}\mathbf{y}]_k = \sqrt{\rho_{\text{ul}}}\mathbf{a}_k^{\text{H}}\mathbf{Z}\mathbf{D}_{\gamma}^{1/2}\mathbf{D}_{\eta}^{1/2}\mathbf{q} + \mathbf{a}_k^{\text{H}}\left(\mathbf{w} - \sqrt{\rho_{\text{ul}}}\tilde{\mathbf{G}}\mathbf{D}_{\eta}^{1/2}\mathbf{q}\right), \quad (\text{H-31})$$

where \mathbf{a}_k is the k th column of \mathbf{A} .

- (a) Using the results in Section 2.3.5, show that an achievable rate for the k th terminal is

$$R_k = \text{E} \{\log_2(1 + \omega_k)\}, \quad (\text{H-32})$$

where

$$\omega_k = \frac{\rho_{\text{ul}} \gamma_k \eta_k |\mathbf{a}_k^H \mathbf{z}_k|^2}{\rho_{\text{ul}} \sum_{k' \neq k}^K \gamma_{k'} \eta_{k'} |\mathbf{a}_k^H \mathbf{z}_{k'}|^2 + \left(1 + \rho_{\text{ul}} \sum_{k'=1}^K (\beta_{k'} - \gamma_{k'}) \eta_{k'}\right) \|\mathbf{a}_k\|^2}. \quad (\text{H-33})$$

- (b) Find \mathbf{a}_k which maximizes ω_k .
- (c) Let \mathbf{A}_{mmse} be the minimizer, with respect to \mathbf{A} , of the mean-square error between the transmitted signal, \mathbf{q} , and its estimate,

$$\mathbb{E} \left\{ \|\mathbf{q} - \mathbf{A}^H \mathbf{y}\|^2 \middle| \mathbf{Z} \right\}. \quad (\text{H-34})$$

Show that \mathbf{a}_k in (b) is the k th column of \mathbf{A}_{mmse} .

3.14 Upper limit on the number of antennas. ^②

Consider an uplink massive MIMO system where a single-antenna terminal transmits signals to the M -antenna base station. The distance between the terminal and the base station is d . We assume that the base station antenna array is disc-shaped with $\lambda/2$ -spaced elements.

- (a) Let d_f be the Fresnel range. Show that

$$d_f = \frac{2D^2}{\lambda}, \quad (\text{H-35})$$

where D is the diameter of the array.

- (b) Determine the Fresnel range as a function of M .
- (c) When the terminal lies inside the Fresnel region (it is in the near-field region), the beamforming gain increases at a lesser rate than M . Find the maximum number of antennas to avoid operating in this Fresnel region for 1.9 GHz operation, and $d = 500$ meters.
- (d) At the Fresnel limit, what fraction of power is intercepted by the array? Find the result for 1.9 GHz operation, and $d = 500$ meters.
- (e) From (3.29), we can see that on the uplink, by increasing M the base station can get more and more power, eventually exceeding the power transmitted by the terminal. Is this true? (Hint: use the result in (c).)

Problems to Chapter 4

4.1 MMSE estimation.^①

Derive (4.3) from first principles.

4.2 Multi-cell operation with frequency reuse for both pilots and data.^②

Consider multi-cell Massive MIMO, and suppose that the same frequency reuse is applied to the data and to the pilots. (All cells that are allocated the same frequency band use the same orthogonal pilot sequences.) Derive uplink and downlink achievable rates for maximum-ratio and zero-forcing processing by appropriately modifying the derivations in Chapter 4. Summarize the results in a table, similar to Table 4.1.

4.3 Multi-cell operation with non-synchronous pilots.^②

Consider a multi-cell Massive MIMO system with two cells. Assume that during the time when K terminals in cell 1 send K orthogonal pilot sequences, the K terminals in cell 2 send data. Derive an uplink achievable rate for the k th terminal in cell 1 for maximum-ratio and zero-forcing processing. Relate to the calculation in Section 4.4.3.

4.4 Alternative multi-cell capacity bound.^②

Consider the multi-cell Massive MIMO uplink with maximum-ratio processing. The processed received signal at the home (l th) cell base station is (4.14).

(a) Using the bounding technique in Section 2.3.5, show that an achievable rate for the k th terminal in the home cell is given by

$$R_{lk} = \mathbb{E} \{ \log_2 (1 + \omega_{lk}) \}, \quad (\text{H-36})$$

where

$$\omega_{lk} = \frac{\rho_{\text{ul}} \gamma_{lk}^l \eta_{lk} \|\mathbf{z}_k^l\|^2}{\rho_{\text{ul}} \sum_{l' \in \mathcal{P}_l \setminus \{l\}} \gamma_{l'k}^{l'} \eta_{l'k} \|\mathbf{z}_k^{l'}\|^2 + \rho_{\text{ul}} \sum_{l' \in \mathcal{P}_l} \sum_{k' \neq k} \gamma_{l'k'}^{l'} \eta_{l'k'} \left| \frac{\mathbf{z}_k^l \mathbf{z}_{k'}^{l'}}{\|\mathbf{z}_k^l\|} \right|^2} + T_1, \quad (\text{H-37})$$

and where

$$T_1 = \rho_{\text{ul}} \sum_{l' \in \mathcal{P}_l} \sum_{k'=1}^K (\beta_{l'k'}^{l'} - \gamma_{l'k'}^{l'}) \eta_{l'k'} + \rho_{\text{ul}} \sum_{l' \notin \mathcal{P}_l} \sum_{k'=1}^K \beta_{l'k'}^{l'} \eta_{l'k'} + 1.$$

(b) Using Jensen's inequality (see Section C.1), show that

$$R_{lk} \geq \log_2 (1 + \text{SINR}_{lk}), \quad (\text{H-38})$$

where

$$\text{SINR}_{lk} = \frac{(M-1)\rho_{\text{ul}}\gamma_{lk}^l\eta_{lk}}{1 + \rho_{\text{ul}}\sum_{l' \in \mathcal{P}_l} \sum_{k'=1}^K \beta_{l'k'}^l \eta_{l'k'} - T_2 + \rho_{\text{ul}}\sum_{l' \notin \mathcal{P}_l} \sum_{k'=1}^K \beta_{l'k'}^l \eta_{l'k'} + (M-1)\rho_{\text{ul}}\sum_{l' \in \mathcal{P}_l \setminus \{l\}} \gamma_{l'k}^l \eta_{l'k}}, \quad (\text{H-39})$$

and where

$$T_2 = \rho_{\text{ul}} \sum_{l' \in \mathcal{P}_l} \gamma_{l'k}^l \eta_{lk}.$$

(c) Compare the bounds (H-38) and (4.18).

4.5 Multi-cell processing for suppression of non-coherent interference. ^②

Consider the multi-cell Massive MIMO uplink, with the pilot and payload signals received at the home cell base station given by (4.1) respectively (4.8).

- (a) Derive a zero-forcing receiver that suppresses non-coherent interference from non-contaminating cells. Discuss its limitations. How does the coherent beamforming gain of this receiver scale with M , K and the pilot reuse?
- (b) Derive the MMSE estimates, $\{\hat{\mathbf{G}}_{l'}^l\}$, of $\{\mathbf{G}_{l'}^l\}$.
- (c) Pretend that the home base station only has access to the channel estimates within the home cell, $\{\hat{\mathbf{G}}_l^l\}$. Derive the MMSE receiver. (Strictly speaking, this receiver is truly the MMSE receiver only in the fictitious case that the contaminating cells – including the home cell – are active in the pilot phase, but all cells are active in the data transmission phase.)
- (d) Derive the MMSE receiver, assuming that the home base station has access to all channel estimates, $\{\hat{\mathbf{G}}_{l'}^l\}$, obtained in (a).
- (e) Which receiver do you expect has the best performance, the one in (a), (c) or (d)? Why do you expect that MMSE is superior to zero-forcing here?

Problems to Chapter 5

5.1 Power control for maximum sum rate.^②

Consider the downlink in a single cell with maximum-ratio processing. Solve the max-sum rate problem

$$\max_{\eta_k \geq 0} \sum_{k=1}^K C_{\text{inst.},k}^{\text{mr,dl}} \quad (\text{H-40})$$

Also determine the conditions under which all terminals receive a non-zero rate.

Hint: reformulate the problem such that it becomes formally identical to a waterfilling problem.

5.2 Effects of providing service to an additional terminal.^{②③}

Consider a single cell with, nominally, $K + 1$ terminals. Generate large-scale fading coefficients by simulating uniformly random locations in an annulus with inner radius 0.1 and outer radius 1.0, a path loss exponent of 4, and lognormal fading with standard deviation 8 dB. In order to “tame” the lognormal distribution of the shadow fading,¹ drop the terminal with the smallest β_k from service, such that K terminals remain.

Consider maximum-ratio processing on the uplink, with a median SNR of -3 dB at the cell border, $M = 100$ base station antennas and $K = 10$ terminals receiving service.

- Plot the cumulative density function of the per-terminal rate with max-min power control. (Generate at least 1000 independent, random realizations of the network.)
- Suppose we want to give service to one more terminal, in addition to the K terminals, that is randomly located in the annulus and has a large-scale fading coefficient with a lognormal distribution truncated at ± 15 dB (standard deviation as above, 8 dB). Plot the cumulative density function of the max-min rate after adding this terminal to service. Discuss the result.
- Redo (a)–(b) without shadow fading. Discuss the result.

5.3 Relative performance of zero-forcing and maximum-ratio processing.^②

Consider a single-cell system. The effective SINR for the k th terminal, SINR_k , is given in Table 3.1. Show that:

- For the uplink, $\text{SINR}_k^{\text{zf,ul}} > \text{SINR}_k^{\text{mr,ul}}$ if and only if

$$\text{SINR}_k^{\text{zf,ul}} > \frac{K \gamma_k \eta_k}{\sum_{k'=1}^K \gamma_{k'} \eta_{k'}} \quad (\text{H-41})$$

¹In practice, in a multi-cell environment, the presence of macro-diversity would yield a qualitatively similar truncation effect on the lognormal distribution’s tail. See also Exercise 6.2.

(b) For the downlink, $\text{SINR}_k^{\text{zf,dl}} > \text{SINR}_k^{\text{mr,dl}}$ if and only if

$$\text{SINR}_k^{\text{zf,dl}} > \frac{K\eta_k}{\sum_{k'=1}^K \eta_{k'}}. \quad (\text{H-42})$$

5.4 Massive MIMO multicasting.^③

Consider multicasting operation in a single cell. There are N simultaneous TV programs being transmitted and the n th program has K_n listeners (terminals). Instead of an orthogonal pilot sequence for each terminal, there are a total of N orthogonal pilot sequences, one for each program. The K_n terminals who want the n th program simultaneously transmit the n th pilot sequence on the uplink, to create and exploit pilot contamination deliberately. To handle the near-far effects among each group of terminals, power control is applied to the uplink pilots in addition to the downlink data.

- (a) Derive the per-terminal SINR for the downlink with maximum-ratio processing.
- (b) Find the combination of uplink and downlink power control that yields max-min throughput over all terminals. Also give the max-min throughput.

5.5 Massive MIMO multicasting with zero-forcing.^③

Repeat Problem 5.4, but for zero-forcing processing.

5.6 Optimal bandwidth.^②

Consider the expressions in Table 5.4, and assume that the radiated power is fixed as the bandwidth B increases. Show that there exists an optimal bandwidth B^* such that the per user throughput is maximized.

Problems to Chapter 6

6.1 Points in a hexagon.^①

Suggest a method, different from that in Appendix G, for the generation of uniformly distributed points in a hexagon.

6.2 Effective distribution of shadow fading in multi-cell environment.^{②③}

Consider a multi-cell system, modeled as a hexagonal topology with a central cell and two rings of surrounding cells (that is, 19 cells in total). The cell radius (vertex-to-vertex distance) is 1, and the base station in the central cell is located at the origin such that there is a vertex at the coordinate (1,0). Only large-scale fading is of concern, and propagation is modeled via path loss with exponent 4 and lognormal shadow fading with standard deviation 8 dB. The shadow fading is uncorrelated between base stations.

Consider a terminal located at the coordinate (0, x). Plot the empirical cumulative distribution of:

- (i) the large-scale fading coefficient to the geographically closest base station,
- (ii) the largest of the large-scale fading coefficients to any of the 19 base stations,
- (iii) the second-largest of the large-scale fading coefficients to any of the base stations, and
- (iv) the fifth-largest of the large-scale fading coefficients to any of the base stations.

Generate one plot for $x = 0.1$ and one for $x = 0.95$. Discuss the result in the context of the base station assignment algorithm in Section 6.2.2.

6.3 Terminal-to-base station association.^①

Consider a given terminal and let SINR_l be its effective downlink SINR when served by base station l . Write this SINR as,

$$\text{SINR}_l = \frac{[\text{received signal power}]_l}{[\text{total received power}]_l - [\text{received signal power}]_l}. \quad (\text{H-43})$$

Show that the choice of base station that maximizes $[\text{received signal power}]_l$ also maximizes SINR_l . Discuss the implications.

6.4 Terminal-to-base station association, continued.^①

Provide an example of a scenario where a terminal's downlink and uplink are best served by different base stations.

6.5 Effects of pilot contamination.^①

Compare the effects of pilot contamination on the coherent beamforming gain and on the coherent interference. Explain why the most significant effect of pilot contamination typically is the reduction of the coherent beamforming gain.

6.6 Computational power consumption.^②

Assume an OFDM implementation of Massive MIMO, and that the computation of the zero-forcing decoder/precoder is performed by QR factorization.

- (a) Estimate the number of arithmetic operations per coherence interval required for the computation of the zero-forcing beamformer, the computation of linear precoding/decoding, and the computation of the channel estimation. Assume that the zero-forcing decoder/precoder is computed once, by QR factorization, requiring MK^2 operations.
- (b) Consider the provision of service to K terminals, where each one uses orthogonal uplink pilots and $\tau_p = K$. Show that when the uplink pilot overhead is less than 50%, the computation of the linear precoding/decoding requires more operations than the computation of the zero-forcing beamformer, and the channel estimation.
- (c) Assume that channel estimates are computed within $T_{\text{slot}}\tau_p/(N_{\text{smooth}}N_{\text{slot}})$ seconds, and that precoding/decoding is done within $T_{\text{slot}}(N_{\text{smooth}}N_{\text{slot}} - \tau_p)/(N_{\text{smooth}}N_{\text{slot}})$ seconds. Furthermore, assume that the computational energy efficiency is 400 GOPS/W,² $T_{\text{slot}}B_c = N_{\text{smooth}}N_{\text{slot}}$, and $\tau_p = K$. Estimate the power consumption of the Massive MIMO zero-forcing beamforming, the precoding/decoding processing, and the channel estimation for the following scenarios:
 - (i) The three scenarios summarized in Tables 6.1, 6.2, and 6.3.
 - (ii) An “extreme Massive MIMO” application with $M = 10^6$, $K = 10^5$, $B = 20$ MHz, $N_{\text{smooth}}N_{\text{slot}} = 3K$ and $B_c = 210$ kHz.
- (d) Consider the energy efficiency in terms of bits transmitted per Joule (per cell) spent on the precoding/decoding computation. Discuss the scaling of this efficiency when M and K simultaneously increase.

²C. Desset and B. Debaillie, “Massive MIMO for energy-efficient communications,” in *Proc. of EuMC*, October 2016.

Problems to Chapter 7

7.1 Point-to-Point MIMO in line-of-sight.^③

Consider Point-to-Point MIMO in free space, with uniform linear arrays at the transmitter and receiver, with M_1 respectively M_2 elements and antenna spacing d_1 respectively d_2 . The arrays are aligned in parallel; the separation between the arrays is D ; $D \gg d_1$ and $D \gg d_2$.

- For what d_1 and d_2 does the channel have rank one?
- For what d_1 and d_2 does the channel have full rank?

Make first-order approximations as appropriate.

7.2 MMSE detection.^①

Consider the error variance of the MMSE detection, see (7.7):

$$\text{Var} \{x_k | \mathbf{y}\} = \left[\left(\mathbf{I}_K + \rho_{\text{ul}} \mathbf{G}^H \mathbf{G} \right)^{-1} \right]_{kk}. \quad (\text{H-44})$$

Show algebraically that

$$\text{Var} \{x_k | \mathbf{y}\} \geq \frac{1}{1 + \rho_{\text{ul}} \|\mathbf{g}_k\|^2}. \quad (\text{H-45})$$

Hint: Formulas for the inverse of a partitioned matrix are useful.

7.3 Orthogonality of beams in line-of-sight.^①

Show that $\{\mathbf{g}_k\}$ in (7.19) satisfy (7.1).

7.4 Uniform rectangular array in isotropic scattering.^{①③}

Consider a uniform rectangular array with $\lambda/2$ spacing in isotropic scattering, using the sinc-correlation model. Compute numerically, and illustrate the behavior of, the singular value distribution of the channel matrix \mathbf{G} for $M = 100$ and $K = 10$. Discuss the result, and compare with the case of a uniform linear array. Does the correlation between diagonally neighboring elements in the rectangular array make an appreciable difference?

7.5 Favorable propagation for Point-to-Point MIMO.^②

Show by example that for a Point-to-Point MIMO channel, assuming that both the transmitter and receiver have perfect CSI and under a constraint on the norms of the columns of the channel matrix, mutual orthogonality between the channel vectors does not necessarily represent the most favorable scenario.

7.6 Derivation of formulas in Table 7.1.^②

Derive the following formula found in Table 7.1, for $k \neq k'$:

$$\frac{1}{M^2} \text{Var} \{ |\mathbf{g}_k^H \mathbf{g}_{k'}|^2 \} = \begin{cases} \frac{M+2}{M}, & \text{for Rayleigh fading,} \\ \frac{(M-1)M(2M-1)}{3M^2}, & \text{for UR-LoS.} \end{cases} \quad (\text{H-46})$$

7.7 Properties of keyhole channel.^②

Consider the keyhole channel, modelled as,

$$\mathbf{g}_k = \nu_k \mathbf{z}_k, \quad (\text{H-47})$$

where $\nu_k \sim \text{CN}(0, 1)$ and $\mathbf{z}_k \sim \text{CN}(0, \mathbf{I}_M)$ are independent. Furthermore, assume that \mathbf{g}_k and $\mathbf{g}_{k'}$ are independent for $k \neq k'$. Show that

- (a) $\frac{1}{M} \mathbf{g}_k^H \mathbf{g}_{k'} \rightarrow 0$, as $M \rightarrow \infty$, $k \neq k'$.
- (b) $\frac{1}{M} \|\mathbf{g}_k\|^2 \xrightarrow{d} \exp(1)$, as $M \rightarrow \infty$.
- (c) $\text{Var} \left\{ \frac{1}{M} \mathbf{g}_k^H \mathbf{g}_{k'} \right\} = \frac{1}{M}$, for $k \neq k'$.
- (d) $\text{Var} \left\{ \frac{1}{M} \|\mathbf{g}_k\|^2 \right\} = 1 + \frac{2}{M}$.

Discuss the results.

7.8 Keyhole propagation.^{②③}

Consider the uplink and suppose propagation is modeled by a keyhole channel with N keyholes. Plot the (empirical) cumulative density function of the sum capacity with $M = 100$ and $K = 10$, for some different N . You can assume that all path losses are equal to unity. Discuss the result, and compare to the case of independent Rayleigh fading.

7.9 Isotropically random unit vector.^③

An isotropically-random unit vector has no preferred direction. Formally, a unit vector, $\boldsymbol{\phi}$, is isotropically random if its probability distribution is invariant to multiplication by an unrelated unitary matrix. That is, if $\boldsymbol{\phi}^H \boldsymbol{\phi} = 1$, then $\boldsymbol{\phi}' = \mathbf{\Theta} \boldsymbol{\phi}$ has the same distribution as does $\boldsymbol{\phi}$, for any unitary matrix $\mathbf{\Theta}$.

- (a) Let \mathbf{x} be an $M \times 1$ vector consisting of i.i.d. $\text{CN}(0, 1)$ random variables. Define a unit vector,

$$\boldsymbol{\phi} = \frac{\mathbf{x}}{\|\mathbf{x}\|}. \quad (\text{H-48})$$

Prove that $\boldsymbol{\phi}$ is isotropically random.

- (b) Consider now the real case where $M = 3$ and \mathbf{x} consists of i.i.d. $N(0, 1)$ random variables. Let

$$\boldsymbol{\phi} = \frac{\mathbf{x}}{\|\mathbf{x}\|}. \quad (\text{H-49})$$

- (i) Prove that ϕ_1 (or any other component of $\boldsymbol{\phi}$) is uniformly distributed on $[-1, 1]$.
(ii) Prove that $\boldsymbol{\phi}$ has the equivalent representation,

$$\boldsymbol{\phi} = \left[\sqrt{1-u^2} \cos(\alpha) \quad \sqrt{1-u^2} \sin(\alpha) \quad u \right]^T, \quad (\text{H-50})$$

where u is uniformly distributed on $[-1, 1]$, and α is uniformly distributed on $[-\pi, \pi]$.

7.10 Propagating, isotropic, random field.^③

- (a) Define Cartesian coordinates

$$[x \quad y \quad z] = \left[\sqrt{1-u^2} \cos \alpha \quad \sqrt{1-u^2} \sin \alpha \quad u \right]. \quad (\text{H-51})$$

Show that on the surface of a sphere of radius R , and incremental patch of area, expressed in terms of u and α , is

$$dA = R^2 du d\alpha. \quad (\text{H-52})$$

- (b) A propagating complex-valued scalar field satisfies the wave equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) h(t, x, y, z), \quad (\text{H-53})$$

or in the frequency Fourier domain, the Helmholtz equation,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \kappa^2 \right) h(\omega, x, y, z), \quad (\text{H-54})$$

where

$$\kappa = \frac{\omega}{c} = \frac{2\pi}{\lambda}. \quad (\text{H-55})$$

For convenience, we drop the ω -dependence. Show that a solution to the Helmholtz equation is a plane wave,

$$h(x, y, z) = e^{i(\kappa_x x + \kappa_y y + \kappa_z z)}, \quad \forall \kappa_x^2 + \kappa_y^2 + \kappa_z^2 = \kappa^2. \quad (\text{H-56})$$

- (c) We can generate a propagating random field as a superposition of plane waves, weighted by independent zero-mean complex Gaussian random variables. In particular, consider

$$h(x, y, z) = \frac{1}{\sqrt{4\pi}} \int_{-1}^1 du \int_0^{2\pi} d\alpha W(u, \alpha) e^{i\kappa[\sqrt{1-u^2}(x \cos \alpha + y \sin \alpha) + uz]}, \quad (\text{H-57})$$

where $W(u, \alpha)$ is a zero-mean, circularly-symmetric complex white-noise random process, such that

$$\mathbb{E} \{W(u + u', \alpha + \alpha') W(u', \alpha')^*\} = \delta(u) \delta(\alpha). \quad (\text{H-58})$$

Prove that each term in the integral in (H-57) is a plane wave. Prove that the random field (H-57) is spatially stationary by obtaining an integral expression for its spatial autocorrelation,

$$\mathbb{E} \{h(x + x', y + y', z + z') h(x', y', z')^*\} = C_h(x, y, z). \quad (\text{H-59})$$

- (d) Prove that the random field is isotropic by showing that the autocorrelation function (H-59) depends only on $R = \sqrt{x^2 + y^2 + z^2}$.
- (e) Obtain an explicit formula for the autocorrelation function.
- (f) Show directly that the explicit autocorrelation function satisfies the Helmholtz equation.

7.11 Diffusion process.^③

Consider the diffusion process, evaluated on the plane $z = 0$. This process has autocorrelation function

$$C_h(x, y) = \mathbb{E} \{h(x + x', y + y') h(x', y')^*\} = \frac{\sin(\kappa \sqrt{x^2 + y^2})}{\kappa \sqrt{x^2 + y^2}}. \quad (\text{H-60})$$

Find the power density spectrum

$$S_h(\kappa_x, \kappa_y) = \int \int dx dy C_h(x, y) e^{-i(\kappa_x x + \kappa_y y)} \quad (\text{H-61})$$

by starting with the integral expression for the autocorrelation

$$C_h(x, y) = \frac{1}{4\pi} \int_{-1}^1 du \int_0^{2\pi} d\alpha e^{i\kappa \sqrt{1-u^2}(x \cos \alpha + y \sin \alpha)}, \quad (\text{H-62})$$

and changing the variables of integration to obtain the equivalent expression

$$C_h(x, y) = \frac{1}{4\pi} \int \int d\kappa_x d\kappa_y S_h(\kappa_x, \kappa_y) e^{i(\kappa_x x + \kappa_y y)}. \quad (\text{H-63})$$

7.12 Hexagonal versus Cartesian sampling.^③

This problem establishes that antenna arrays based on hexagonal sampling can be more efficient than those based on Cartesian sampling. It will be useful to recall that the Fourier transform of an infinite sequence of evenly spaced impulses (Dirac delta functions) is another infinite sequence of impulses:

$$\int dx e^{-i\kappa_x x} \sum_{m=-\infty}^{\infty} \delta(x - am) = \frac{2\pi}{a} \sum_{p=-\infty}^{\infty} \delta\left(\kappa_x - \frac{2\pi p}{a}\right). \quad (\text{H-64})$$

Hence a propagating field, $h(x)$, sampled at intervals of a , has the following Fourier transform:

$$\begin{aligned} \int dx e^{-i\kappa_x x} \sum_{m=-\infty}^{\infty} \delta(x - am) h(x) &= \frac{1}{2\pi} \left[\frac{2\pi}{a} \sum_{p=-\infty}^{\infty} \delta\left(\kappa_x - \frac{2\pi p}{a}\right) \right] * H(\kappa_x) \\ &= \frac{1}{a} \sum_{p=-\infty}^{\infty} H\left(\kappa_x - \frac{2\pi p}{a}\right), \end{aligned} \quad (\text{H-65})$$

where “*” denotes the convolution operation, and $H(\kappa_x)$ is the Fourier transform of $h(x)$.

Since $H(\kappa_x)$ is bandlimited with support $\kappa_x \in \left[-\frac{2\pi}{\lambda}, \frac{2\pi}{\lambda}\right]$, it follows that the periodic replication of $H(\kappa_x)$ causes no ambiguity as long as $\frac{2\pi}{a} \geq 2\frac{2\pi}{\lambda}$, or $a \leq \frac{\lambda}{2}$. This is the Nyquist sampling theorem.

Now suppose that a propagating field is sampled hexagonally in the plane $z = 0$, where the distance between each sample and its nearest neighbors is equal to a , where the sampling function is

$$\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta\left(x - am - \frac{a}{2}n\right) \delta\left(y - \frac{a\sqrt{3}}{2}n\right). \quad (\text{H-66})$$

- (a) Show that the 2D Fourier transform of (H-66) is also an infinite sequence of impulses in (κ_x, κ_y) on a hexagonal grid, and that the spacing between each impulse and its nearest neighbors is equal to $\frac{\pi}{a\sqrt{3}}$.
- (b) A 2D propagating field has spectral support

$$(\kappa_x, \kappa_y) : \kappa_x^2 + \kappa_y^2 \leq \left(\frac{2\pi}{\lambda}\right)^2.$$

What is the minimum hexagonal sampling interval, a , needed to avoid aliasing?

- (c) A rectangular array has the area A . If Cartesian Nyquist sampling is employed, then the number of antennas is

$$M \approx \frac{4A}{\lambda^2}. \quad (\text{H-67})$$

How many antennas are required for hexagonal Nyquist sampling?

- (d) Can you think of any other advantage of hexagonal sampling over Cartesian sampling?

7.13 Normalized inner product between i.i.d. Gaussian vectors.^③

Let $\{\mathbf{g}_1, \mathbf{g}_2, \dots, \mathbf{g}_K\}$ be statistically independent channel vectors, each consisting of M i.i.d. $\text{CN}(0, 1)$ elements.

- (a) Find the distribution of the normalized inner product between \mathbf{g}_k and $\mathbf{g}_{k'}$,

$$\frac{|\mathbf{g}_k^H \mathbf{g}_{k'}|^2}{\|\mathbf{g}_k\|^2 \|\mathbf{g}_{k'}\|^2}. \quad (\text{H-68})$$

- (b) Find an upper bound on the probability that the maximum normalized inner product between any two distinct channel vectors is greater than some threshold γ , i.e.,

$$\mathbb{P}\left(\max_{k \neq k'} \frac{|\mathbf{g}_k^H \mathbf{g}_{k'}|^2}{\|\mathbf{g}_k\|^2 \|\mathbf{g}_{k'}\|^2} \geq \gamma\right). \quad (\text{H-69})$$

- (c) Numerically evaluate the tightness of the bound in (b), for $M = 100$ and $K = 20$.

7.14 Uplink in light-of-sight.^②

Consider the uplink with two terminals transmitting to a base station equipped with $M = 2$ antennas. Assume that the channel matrix between the base station and the terminals is

$$\mathbf{G} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{1+x^2}} \\ \frac{1}{\sqrt{2}} & \frac{x}{\sqrt{1+x^2}} \end{bmatrix}, \quad (\text{H-70})$$

and that the base station knows \mathbf{G} perfectly.

Let the signal transmitted from terminal k be $\sqrt{\eta_k \rho} q_k$, where q_k , $\mathbb{E}\{|q_k|^2\} = 1$, is the transmitted symbol, $0 \leq \eta_k \leq 1$ is the power control coefficient, and ρ is the normalized power.

- (a) Assume that both terminals share the same time-frequency resource, and zero-forcing processing is used at the base station. Find the values of $\{\eta_k\}$ that maximize the minimum spectral efficiency (i.e., the optimal power control coefficients in the max-min fairness sense).
- (b) Consider an orthogonal multiple-access scheme where terminal 1 is allocated a fraction α , $0 < \alpha < 1$, of the degrees of freedom, and terminal 2 is allocated the remaining fraction, $1 - \alpha$, of the degrees of freedom. Find the values of $\{\eta_k\}$ and α that maximize the minimum spectral efficiency.
- (c) Compare the spectral efficiencies in (a) and (b) for $x = -1, -0.5, 1.1, \text{ and } 2$, and $\rho = 10$ dB. Discuss the result.

Problems to Appendix A

9.1 Moments of Gaussian variables.^①

Prove (A.22) by expanding z into its real and imaginary parts.

9.2 Moments of Gaussian variables.^①

Let $z \sim \text{CN}(0, 1)$. Compute $E\{z^2\}$ and $E\{z^4\}$.

9.3 Probability density function.^①

Let $z \sim \text{CN}(0, 1)$. Derive the density function of $|z|^2$.

9.4 Moments of Gaussian variables.^①

Let \mathbf{z} be a zero-mean, real-valued Gaussian random vector with independent elements of unit variance. Compute $E\{\|\mathbf{z}\|^4\}$. Compare with the complex-valued case.

9.5 Moments of Gaussian variables.^②

Let $\mathbf{x} \in \mathbb{C}^{M \times 1}$, $\mathbf{y} \in \mathbb{C}^{M \times 1}$, and $\mathbf{W} \in \mathbb{C}^{M \times M}$ be mutually independent. The elements of \mathbf{x} are i.i.d. $\text{CN}(0, \sigma_x^2)$, the elements of \mathbf{y} are i.i.d. $\text{CN}(0, \sigma_y^2)$, and the elements of \mathbf{W} are i.i.d. $\text{CN}(0, \sigma_w^2)$. Compute

$$(a) \ E\{|\mathbf{x}^H \mathbf{y}|^2\}$$

$$(b) \ E\{|\mathbf{x}^H \mathbf{W} \mathbf{y}|^2\}$$

9.6 Moments of Gaussian variables.^③

Let $\mathbf{x} = [x_1, x_2, \dots, x_M]^T$ be a zero-mean circularly symmetric complex Gaussian random vector, $\mathbf{x} \sim \text{CN}(\mathbf{0}, \mathbf{\Lambda})$.

- (a) Consider the moment given by the expectation of the product of M_1 components of \mathbf{x} times the complex conjugate of the product of M_2 components of \mathbf{x} (the components can be reused any number of times),

$$E\left\{\left(x_{l_1} x_{l_2} \cdots x_{l_{M_1}}\right) \left(x_{n_1} x_{n_2} \cdots x_{n_{M_2}}\right)^*\right\}, \quad (\text{H-71})$$

where $l_j \in \{1, 2, \dots, M\}$ and $n_j \in \{1, 2, \dots, M\}$. Show that this moment vanishes if $M_1 \neq M_2$.

- (b) Show that

$$E\{(x_1 x_2)(x_3 x_4)^*\} = E\{x_1 x_3^*\} E\{x_2 x_4^*\} + E\{x_1 x_4^*\} E\{x_2 x_3^*\} \quad (\text{H-72})$$

- (c) Generalize the result (H-72) to any M , by showing that the moment is equal to the sum of $\left(\frac{M}{2}\right)!$ terms, each term consisting of a product of M distinct second-order moments. In particular, we have the following six-term expression:

$$\begin{aligned}
& \mathbb{E} \{ (x_1 x_2 x_3) (x_4 x_5 x_6)^* \} \\
&= \mathbb{E} \{ x_1 x_4^* \} \mathbb{E} \{ x_2 x_5^* \} \mathbb{E} \{ x_3 x_6^* \} + \mathbb{E} \{ x_1 x_4^* \} \mathbb{E} \{ x_2 x_6^* \} \mathbb{E} \{ x_3 x_5^* \} \\
&+ \mathbb{E} \{ x_1 x_5^* \} \mathbb{E} \{ x_2 x_4^* \} \mathbb{E} \{ x_3 x_6^* \} + \mathbb{E} \{ x_1 x_5^* \} \mathbb{E} \{ x_2 x_6^* \} \mathbb{E} \{ x_3 x_4^* \} \\
&+ \mathbb{E} \{ x_1 x_6^* \} \mathbb{E} \{ x_2 x_4^* \} \mathbb{E} \{ x_3 x_5^* \} + \mathbb{E} \{ x_1 x_6^* \} \mathbb{E} \{ x_2 x_5^* \} \mathbb{E} \{ x_3 x_4^* \}.
\end{aligned} \tag{H-73}$$

9.7 Stationarity and circular symmetry.^②

Let $e(t)$ be a real-valued, stationary narrowband Gaussian random process, such that its spectral density is symmetric and has finite support around $\pm f_c$, where f_c is the carrier frequency. Suppose $e(t)$ is constructed from stationary, zero-mean I- and Q-components, $e_I(t)$ and $e_Q(t)$, according to,

$$e(t) = e_I(t) \cos(2\pi f_c t) - e_Q(t) \sin(2\pi f_c t). \tag{H-74}$$

- (a) Using the following notation for any stationary processes $x(t)$ and $y(t)$:

$$\phi_{xy}(T) = \mathbb{E} \{ x(t)y(t+T) \}, \tag{H-75}$$

show that the auto- and cross-covariance functions of $e_I(t)$ and $e_Q(t)$ satisfy the following symmetry conditions:

$$\begin{aligned}
\phi_{e_I e_I}(T) &= \phi_{e_Q e_Q}(T) \\
\phi_{e_I e_Q}(T) &= -\phi_{e_Q e_I}(T)
\end{aligned} \tag{H-76}$$

- (b) Show that

$$\begin{aligned}
\phi_{e_I e_I}(0) &= \phi_{e_Q e_Q}(0) \\
\phi_{e_I e_Q}(0) &= 0.
\end{aligned} \tag{H-77}$$

- (c) Let

$$\tilde{e}(t) = e_I(t) + ie_Q(t) \tag{H-78}$$

be the complex baseband equivalent of $e(t)$. Relate (H-77) to the concept of circular symmetry.

Problems to Appendix B

10.1 Householder transformation.^②

In Section B.2:

- (a) What can be said about the eigenvalues of \mathbf{H} ?
- (b) Show that \mathbf{H} is unitary.
- (c) Verify (B.8).

10.2 Wishart distribution.^②

Let \mathbf{Z} be an $M \times K$, $M \geq K$, matrix whose elements are i.i.d. $\text{CN}(0, 1)$ random variables. Then the elements of the $K \times K$ matrix $\mathbf{Z}^H \mathbf{Z}$ have the joint complex Wishart distribution of dimension K with M degrees of freedom, denoted $\mathbf{Z}^H \mathbf{Z} \sim \text{CW}(K, M, \mathbf{I}_M)$. (Here \mathbf{I}_M is the covariance matrix of any column of \mathbf{Z} .) Let $\mathbf{\Phi}$ be a $K \times L$, $L \leq K$, unitary matrix that is uncorrelated to \mathbf{Z} . Prove that the “double inverse”, $\left(\mathbf{\Phi}^H \left(\mathbf{Z}^H \mathbf{Z} \right)^{-1} \mathbf{\Phi} \right)^{-1}$, has distribution $\text{CW}(L, M + L - K, \mathbf{I}_{M+L-K})$. Note that this result, for the special case of $L = 1$, yields the identity (B.2).

10.3 Property of a Wishart distributed matrix.^③

Let \mathbf{Z} be an $M \times K$, $M \geq K$, matrix whose elements are i.i.d. $\text{CN}(0, 1)$ random variables, and let \mathbf{s} be an $M \times 1$ vector whose entries are positive real. Define a $K \times K$ matrix \mathbf{A} by

$$\mathbf{A} = \mathbb{E} \left\{ \left(\mathbf{Z}^H \mathbf{D}_s \mathbf{Z} \right)^{-1} \right\}. \quad (\text{H-79})$$

Prove that \mathbf{A} is equal to an undesignated constant times the identity matrix.

Problems to Appendix C

11.1 Fading versus deterministic channel.^①

Compare the model for a deterministic channel with Gaussian noise in Section 2.3.1, with that of a fading channel with Gaussian noise in Section 2.3.3. Assume that $E\{|g|^2\} = 1$.

- Show that the received power is the same in both cases.
- Show analytically that C in (2.42) cannot exceed C in (2.38). Numerically quantify the gap between them. Comment on the result.

11.2 Point-to-Point MIMO without transmitter CSI.^②

Consider the deterministic Point-to-Point MIMO channel (for the uplink, say) without CSI at the transmitter, so that (1.1) holds.

- Consider the limit when $\rho \rightarrow 0$. How does C scale with K ? How does C scale with M ? At low SNR, are multiple antennas useful, and if so, under what circumstances?

For the sake of argument, assume that \mathbf{G} is such that $|g_k^m| = 1$ (but make no other assumptions). Also, justify this assumption.

- Consider the limit when $\rho \rightarrow \infty$. How does C scale with K ? How does C scale with M ? At high SNR, are multiple antennas useful, and if so, under what circumstances?

For this question, consider two different assumptions on \mathbf{G} : (i) \mathbf{G} is such that $|g_k^m| = 1$ and has rank one; (ii) \mathbf{G} is such that $|g_k^m| = 1$ and all its singular values are equal.

Hint: expand $\mathbf{G}\mathbf{G}^H$ in terms of its eigenvalue decomposition. Treat the cases of $M > K$ and $M \leq K$ separately.

11.3 Outage probability of Point-to-Point MIMO.^②

Consider the capacity of a deterministic Point-to-Point MIMO channel without CSI at the transmitter; see (1.1).

- Show that for any $\rho > 0$,

$$\frac{C}{\log_2(e)} \leq \rho \text{Tr}\{\mathbf{G}^H \mathbf{G}\} = \rho \sum_{m=1}^M \sum_{k=1}^K |g_{mk}|^2 \leq MK\rho \max_{m,k} |g_k^m|^2 \quad (\text{H-80})$$

- Suppose \mathbf{G} is drawn randomly. Consider the probability of outage, that is, $P(C < R)$ for given R . Find a lower bound on this probability. In i.i.d. Rayleigh fading, with $\{g_k^m\}$ independent $\text{CN}(0, 1)$, how does the probability of outage depend on ρ ? What order of spatial diversity is achieved? Do you know of any practical methods of achieving this diversity?

11.4 Point-to-Point MIMO with transmitter and receiver CSI.^②

Consider the deterministic Point-to-Point MIMO channel, and suppose the transmitter and receiver have CSI so that (C.28) applies. Compare C in (C.28) with C in (C.30) for low respectively high SNR, under the two assumptions that

- (a) \mathbf{G} is such that $|g_k^m| = 1$ and has rank one
- (b) \mathbf{G} is such that $|g_k^m| = 1$ and all its singular values are equal.

Also, give the optimal transmit covariance and interpret the result. In what operating regimes is the transmit CSI useful?

11.5 Point-to-point MIMO, uplink versus downlink.^①

Compare the Point-to-Point MIMO capacity without transmit CSI, for the uplink, (1.1), and for the downlink, (1.2). Under what conditions are these two capacities equal?

11.6 Channel rank.^②

Consider an i.i.d. Rayleigh fading Point-to-Point MIMO channel. Let \mathbf{G} be the $M \times K$ channel matrix. What is the probability that \mathbf{G} has linearly dependent columns?

11.7 Multiuser MIMO uplink.^②

Consider the capacity of the uplink (multiple-access) multi-user MIMO channel with perfect CSI everywhere, (1.3). Show that the sum-capacity increases without bound if $K \rightarrow \infty$. For the sake of argument, assume that \mathbf{G} is such that $|g_k^m| = 1$ (but make no other assumptions on \mathbf{G}).

11.8 Multiuser MIMO downlink.^①

Consider the sum-capacity of the downlink (broadcast) channel, (1.4), assuming that CSI is available everywhere. Suppose $M \geq K$.

- (a) If $K = 1$, what is the optimal transmit strategy and what is the resulting capacity? Compare with the capacity of a Point-to-Point MIMO channel with CSI at the transmitter, respectively no CSI at the transmitter. For the sake of argument, assume that \mathbf{G} is such that $|g_k^m| = 1$.
- (b) If $K > 1$, and $\mathbf{G}^H \mathbf{G} = \mathbf{D}_\beta$ for some vector β , what is the optimal transmit strategy and what is the resulting sum-capacity?

11.9 Multiuser MIMO downlink, without transmitter CSI.^①

Consider the downlink (broadcast) channel, with M antennas at the base station and K terminals but without CSI at the base station. Propose a transmission scheme that achieves $\log_2(1 + \rho)/K$ bpcu/terminal. Are multiple base station antennas useful?

For the sake of argument, assume that \mathbf{G} is such that $|g_k^m| = 1$ (but no other assumptions). Also you can assume that the channel coherence is long enough that the terminals have perfect CSI.

11.10 Jensen's inequality.^②

Give an induction proof of Jensen's inequality, (C.1), for the case of a discrete probability distribution.

11.11 Notation.^①

Let x and y be random variables, possibly statistically dependent. With the notation used in the book, which of the following quantities are random and which are deterministic: $E\{x\}$, $E\{x|y\}$, $\text{Var}\{x\}$, $\text{Var}\{x|y\}$, $h\{x\}$, $h\{x|y\}$, $I\{x; y\}$, $p_x(t)$, $p_x(x)$? Under what circumstances is $E\{\text{Var}\{x|y\}\} = \text{Var}\{x\}$?

11.12 Capacity bound with correlated effective noise.^②

Consider the scalar channel model in Section 2.3.2 but with the modification that x and w are potentially correlated. Show that

$$C \geq \log_2 \left(1 + \frac{|E\{y^*x\}|^2}{E\{|y|^2\} - |E\{y^*x\}|^2} \right). \quad (\text{H-81})$$

11.13 Rigorous derivation of the capacity bound with side information.^③

Prove the step (a) in (C.20), and verify that the expectation of (C.20) with respect to Ω yields (2.46).

11.14 Flawed capacity bound with uncorrelated noise.^②

Consider the fading, point-to-point channel,

$$y = \sqrt{\rho}gx + w, \quad (\text{H-82})$$

where g is the channel gain, x is the transmitted signal, and w is additive noise. We assume that w , g , and x have zero mean, and that x is independent of g . Furthermore, we assume that the receiver knows g , and that w is uncorrelated with x , i.e.,

$$E\{x^*w\} = 0. \quad (\text{H-83})$$

(a) Assess the following claim:

“Since w is uncorrelated with x , we can use the worst-case Gaussian noise argument, and obtain the following capacity bound:

$$C \geq E \left\{ \log_2 \left(1 + \frac{\rho|g|^2}{E\{|w|^2\}} \right) \right\}.” \quad (\text{H-84})$$

- (b) If you found that the claim in (a) is incorrect, give a counterexample – a channel for which C is smaller than the right hand side of (H-84). Also, explain why the claim is incorrect; what condition is missing?

11.15 Flawed capacity bound without CSI at the receiver.^②

Consider again the point-to-point channel in Problem 11.14; however, assume that neither the transmitter nor the receiver knows g .

- (a) Assess the following claim:

“Since $E\{x^*w\} = 0$, the use of (2.44) yields

$$C \geq \log_2 \left(1 + \frac{\rho |E\{g\}|^2}{\rho \text{Var}\{g\} + 1} \right).” \quad (\text{H-85})$$

Is this claim correct? Why, or why not?

- (b) If the claim in (a) is incorrect, give a counterexample.

11.16 Multiuser MIMO uplink with orthogonal access.^②

For the uplink (multiple-access) channel in Section C.4.1, consider orthogonal access, that is, terminal 1 transmitting a fraction α of the time and terminal 2 transmitting a fraction $1 - \alpha$.

- (a) Give a formula for the sum capacity with orthogonal access, parameterized in terms of α .
- (b) Can orthogonal access be optimal in terms of sum-capacity? If so, why and under what conditions?

11.17 Max-min fairness multiple-access.^②

Consider the uplink multiple-access channel (C.32), where \mathbf{G} is perfectly known by the base station, and every terminal transmits at full power, $E\{|x_k|^2\} = 1$.

- (a) Prove the sum-capacity formula (C.34) for arbitrary K by showing that this can be achieved by successive decoding, and subtraction of the decoded signal.
- (b) While the sum-capacity is the same for any ordering of the decoding, the individual capacities do depend on the ordering. Consider the following decoding order:
- First decode the signal, while treating the other $K - 1$ signals as noise, that enjoys the greatest SINR, and then subtract the decoded signal.
 - Among the remaining $K - 1$ signals, decode and subtract the signal that enjoys the greatest SINR, and so on.

Show that the above ordering yields the max-min per-terminal capacity.

11.18 Non-coherent Point-to-Point MIMO.^②

Consider an $M \times K$ Point-to-Point MIMO channel,

$$\mathbf{y} = \sqrt{\rho} \mathbf{G} \mathbf{x}_t + \mathbf{w}_t, \quad (\text{H-86})$$

where \mathbf{G} is a $K \times M$ channel matrix. We assume a block-fading channel model, under which \mathbf{G} is piecewise constant over coherence intervals of τ_c symbols, and takes on a statistically independent realization within each coherence interval. The elements of \mathbf{G} are assumed to be i.i.d. $\text{CN}(0, 1)$. Neither the transmitter nor the receiver knows \mathbf{G} . Channel coding is performed over many coherence intervals. Within each coherence interval, the received signal is related to the transmitted signal as follows,

$$\mathbf{Y} = \sqrt{\rho} \mathbf{G} \mathbf{X} + \mathbf{W}, \quad (\text{H-87})$$

where \mathbf{Y} is $K \times \tau_c$, and \mathbf{X} is $M \times \tau_c$.

- (a) Prove that there is no point in making $M > \tau_c$, i.e., the capacity for $M > \tau_c$ is equal to the capacity for $M = \tau_c$.
- (b) For $M < \tau_c$, prove that the capacity-attaining transmitted signals that are fed to the M antennas are temporally orthogonal.
- (c) What does the above result tell us about the downlink multiuser MIMO channel? What is the limitation of the above result?